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by Gerard van der Laan,
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Intersection Theorems on Polytopes *

Gerard van der Laan, † Dolf Talman, ‡ Zaifu Yang §

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†Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands.

‡Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

§Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

Abstract

Intersection theorems are used to prove the existence of solutions to mathematical programming and game theoretic problems. Well-known intersection theorems are the theorems of Sperner, Knaster-Kuratowski-Mazurkiewicz (KKM), Scarf, Shapley, Ichiishi and Gale on the unit simplex. Recently the intersection result of KKM has been generalized by Ichiishi and Idzik to closed coverings of a compact convex polyhedron, called a polytope. In this paper we formulate a general intersection theorem on the polytope. To do so, we need to generalize the concept of balancedness as is used by Shapley and Ichiishi. The theorem implies most of the results stated above as special cases. First, we show that the theorems of KKM, Sperner, Scarf, Shapley and Ichiishi on the unit simplex and also some theorems of Ichiishi and Idzik on a polytope all satisfy the conditions of our theorem on the polytope. Secondly, the general theorem allows us to formulate the analogs of these theorems on the polytope.

Key words: intersection theorem, unit simplex, polytope, closed covering, balancedness

1 Introduction

Intersection theorems are used to prove the existence of solutions to mathematical programming problems. The probably most well-known intersection theorem is the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma on the unit simplex. This lemma (see Knaster, Kuratowski and Mazurkiewicz [9]) states that n closed subsets covering the $(n-1)$-dimensional unit simplex S^n and satisfying some boundary condition have a nonempty intersection. A related theorem is due to Sperner [16]. The reformulation of this theorem given by Fan [1] and Freidenfelds [2] can also be found in Scarf [14] and is known as Scarf's lemma. Further generalizations of intersection theorems on the unit simplex can be found in Scarf [13], Shapley [15], Gale [5], Freund [4], Ichiishi [6] and Joosten and Talman [8]. Moreover, generalizations of intersection theorems to the cube or simplotope are stated in Freund [4], van der Laan, Talman and Van der Heyden [11], van der Laan and Talman [12] and Talman [17]. These generalizations can be used to prove the existence of a Nash equilibrium in a noncooperative N-person game.

Recently the intersection theory has been generalized by Ichiishi and Idzik [7] to closed coverings of a compact, convex polyhedron, called a polytope. They generalize the KKM lemma and show that Gale's theorem on the unit simplex can be derived from their intersection result on the polytope. In this paper we formulate a general intersection theorem on the polytope. The theorem differs from the results mentioned above by the way in which the boundary conditions are formulated. As far as we know, in all intersection theorems the boundary condition is of the form that every facet of the set under consideration (simplex, simplotope or polytope) is covered by the union of some specified elements of the family of sets forming the covering. In our theorem we give one general condition to be satisfied at every point on the boundary. This results in a theorem which implies most of the results stated above as special cases. In particular we show that the theorems of KKM, Sperner, Scarf, Shapley and Ichiishi on the unit simplex and also some theorems of Ichiishi and Idzik on a polytope all satisfy the boundary condition from our theorem on the polytope and therefore can be derived from our theorem as special cases. Moreover we show that Gale's theorem can be proved by applying our theorem. Our theorem also allows to derive the analogs of the theorems on the unit simplex to the polytope. In particular, we formulate the analogs of the theorems of KKM, Sperner, Shapley and Ichiishi on the polytope. One of these results is closely related to a result given in Ichiishi and Idzik [7].

This paper is organized as follows. Section 2 contains the mathematical preliminaries. Several concepts and some notation are introduced concerning the polytope. Also the concept of balancedness of a collection of vectors is given. In Section 3 we formulate and prove the main result. In Section 4 we show that most of the results on the unit simplex follow as special cases from our main theorem. In Section 5 we derive some special theorems on the polytope. These theorems can be seen as the analogs of the theorems of KKM, Sperner, Shapley and Ichiishi on the unit simplex.

2 Notation and preliminaries

For n an integer, the set of integers $\{1,\ldots,n\}$ is denoted by I_n. For given l, $0 < l \leq n$, let the $n-l$ vectors d^h, $h \in I_{n-l}$ in the n-dimensional space $\mathbf{R}^n$ form an orthogonal basis for the $(n-l)$-dimensional subspace

$$V = \left\{ x \in \mathbf{R}^n \mid x = \sum_{h=1}^{n-l} \nu_h d^h,\ \nu_h \in \mathbf{R} \text{ for } h \in I_{n-l} \right\}.$$

For $l = n$ we define $V = \{\underline{0}\}$. We define

$$V^* = \left\{ x \in \mathbf{R}^n \mid x^\top y = 0, \text{ for all } y \in V \right\}$$

as the l-dimensional subspace V^* orthogonal to V. Let I be a set of at least $l+1$ integers. Then, for given vectors a^i, $i \in I$ in $\mathbf{R}^n$, and real numbers α^i, $i \in I$, and δ_h, $h \in I_{n-l}$, let the polytope P be defined by

$$P = \left\{ x \in \mathbf{R}^n \mid a^{i\top} x \leq \alpha_i,\ i \in I \text{ and } d^{h\top} x = \delta_h,\ h \in I_{n-l} \right\}.$$

Without loss of generality, we assume that P is a simple l-dimensional set in $\mathbf{R}^n$ and that none of the constraints $a^{i\top} x \leq \alpha_i$, $i \in I$, is redundant.

For $T \subset I$, we define

$$F(T) = \left\{ x \in P \mid a^{i\top} x = \alpha_i \text{ for } i \in T \right\}.$$

Observe that $F(\emptyset) = P$. In case $F(T)$ is nonempty, we call $F(T)$ a face of P. We notice that for every face $F(T)$ the dimension of $F(T)$ is equal to $k - |T|$, where $|T|$ denotes the number of elements in T. When, for some $i \in I$, $T = \{i\}$, we call $F(T)$ a facet of P and we denote this facet by F_i. When $|T| = l$ and $F(T)$ is nonempty, $F(T)$ is a vertex of P. Furthermore, let $\mathrm{bnd}(P)$ denote the relative boundary of the set P, i.e., for a point $x \in \mathrm{bnd}(P)$ at least one of the constraints $a^{i\top} x = \alpha_i$, $i \in I$, is binding. For $x \in \mathrm{bnd}(P)$, the set I_x is defined as

$$I_x = \left\{ i \in I \mid a^{i\top} x = \alpha_i \right\},$$

i.e., I_x is the set of indices for which the corresponding constraint is binding at x. Clearly, for any $T \subset I_x$, x belongs to $F(T)$. Moreover, for $T \subset I$, let $A(T)$ be the cone defined by

$$A(T) = \left\{ x \in \mathbf{R}^n \mid x = \sum_{i \in T} \lambda_i a^i + \sum_{h=1}^{n-l} \nu_h d^h,\ \lambda_i \geq 0,\ i \in T \text{ and } \nu_h \in \mathbf{R},\ h \in I_{n-l} \right\}$$

with $A(\emptyset) = \{\underline{0}\}$ when $l = n$, and let $A^*(T)$ be the polar cone of $A(T)$ defined by

$$A^*(T) = \left\{ y \in \mathbf{R}^n \mid y^\top x \leq 0 \text{ for all } x \in A(T) \right\}.$$

By definition we have that $A(T) \cap A^*(T) = \{\underline{0}\}$, for all $T \subset I$.

Finally, for some finite nonempty set J, let be given a collection of vectors c^j, $j \in J$, in $\mathbf{R}^n$. For a nonempty set $T \subset J$, we define

$$C(T) = \left\{ x \in \mathbf{R}^n \mid x = \sum_{j \in T} \mu_j c^j, \ \sum_{j \in T} \mu_j = 1 \text{ and } \mu_j \geq 0, \, j \in T \right\},$$

i.e., $C(T)$ is the convex hull of the vectors c^j, $j \in T$. We assume that $\underline{0} \in C(J)$. We now give the following definition.

Definition 2.1
For some nonempty subset T of J, the collection of vectors $\{c^j \mid j \in T\}$ is balanced when the origin lies in the relative interior of $C(T)$, i.e., if the system of linear equations $\sum_{j \in T} \mu_j c^j = \underline{0}$ has a positive solution μ_j^*, $j \in T$.

Without confusion, we say that the set T is balanced if the corresponding collection of vectors is balanced.

3 Intersection theorem

Given a polytope P and a collection of vectors $\{c^j \mid j \in J\}$ in $\mathbf{R}^n$, take a collection of sets $\{C^j \mid j \in J\}$ being a closed covering of the polytope P. For $x \in P$, we define $J_x = \{j \in J \mid x \in C^j\}$. In the next theorem we give a sufficient condition to guarantee that there is a balanced set of indices in J for which the intersection of sets labelled by these indices is nonempty.

Main Theorem
Let the collection of vectors $\{c^j \mid j \in J\}$ in $\mathbf{R}^n$ be such that $C(J) \cap V = \{\underline{0}\}$ and let $\{C^j \mid j \in J\}$ be a collection of closed sets covering the polytope P such that for every $x \in bnd(P)$,

$$C(J_x) \cap A^*(I_x) \neq \emptyset.$$

Then there exists a balanced set $T^* \subset J$ for which $\cap_{j \in T^*} C^j \neq \emptyset$.

<u>Proof.</u>
Let the polytope P' be defined by

$$P' = \left\{ x \in \mathbf{R}^n \mid a^{i\top} x \leq \alpha_i + 1, \, i \in I, \, d^{h\top} x = \delta_h, \, h \in I_{n-l} \right\}.$$

Clearly, P' is nonempty, convex and compact and P' contains P in its relative interior. For $x \in P'$, let $p(x)$ be the orthogonal projection of x on P. Because of the convexity of P, the function $p: P' \to P$ is continuous. For given $x \in P' \setminus P$, there exist unique nonnegative numbers μ_i, $i \in I_{p(x)}$, such that $x - p(x) = \sum_{i \in I_{p(x)}} \mu_i a^i$. We define $I(x) = \{i \in I_{p(x)} \mid \mu_i > 0\}$. For any sequence of points $(x^1, x^2, \ldots)$ in $P' \setminus P$ with $I(x^h) = I'$ for some I' and for every h, converging to a point $x^* \in P' \setminus P$, it holds that $I(x^*) \subset I'$. Now, let the point-to-set mapping $F: P' \to C(J)$ be defined by

$$F(x) = C(J_x) \text{ if } x \in P,$$

$$F(x) = A^*(I(x)) \cap C(J_{p(x)}) \text{ if } x \in P' \setminus P.$$

Since, for every $x \notin P$, $I(x) \subset I_{p(x)}$ we have that $A(I(x)) \subset A(I_{p(x)})$ and hence $A^*(I_{p(x)}) \subset A^*(I(x))$. So, $C(J_{p(x)}) \cap A^*(I_{p(x)}) \subset C(J_{p(x)}) \cap A^*(I(x))$ and therefore $F(x) \neq \emptyset$ if $x \notin P$, because of the boundary condition. Clearly, since the sets C^j are a covering of P, $F(x) \neq \emptyset$ if $x \in P$. Furthermore, F is upper-hemi continuous because $I(x^*) \subset I'$ for any sequence $\{x^1, x^2, \ldots\}$ of points with $I(x^h) = I'$ for every h, converging to a point $x^* \in P' \setminus P$. Moreover, by definition $F(x)$ is convex and compact. Now, let the point-to-set mapping G from $C(J)$ to the collection of subsets of P' be given by

$$G(y) = \left\{x^* \in P' \mid x^\top y \leq x^{*\top} y \text{ for every } x \in P'\right\},\ y \in C(J).$$

Also G is upper hemi-continuous and for every $y \in C(J)$ the set $G(y)$ is nonempty, convex and compact. Hence, according to Kakutani's fixed point theorem the mapping H from the Cartesian product $P' \times C(J)$ to the collection of subsets of this set, defined by $H(x,y) = G(y) \times F(x)$, $(x,y) \in P' \times C(J)$, has a fixed point on $P' \times C(J)$, i.e., there exists an $(x^*, y^*) \in P' \times C(J)$ such that $x^* \in G(y^*)$ and $y^* \in F(x^*)$. From $x^* \in G(y^*)$ it follows that

$$x^\top y^* \leq x^{*\top} y^*, \text{ for every } x \in P.$$

Consequently, x^* solves the problem

$$\max x^\top y^* \text{ s.t. } a^{i\top} x \leq \alpha_i + 1,\ i \in I,\ d^{h\top} x = \delta_h,\ h \in I_{n-l}.$$

From the solution to the dual of this problem it follows that there exist nonnegative numbers λ_i^*, $i \in I$, and ν_h^*, $h \in I_{n-l}$ satisfying

$$y^* = \sum_{i \in I} \lambda_i^* a^i + \sum_{h \in I_{n-l}} \nu_h^* d^h.$$

From $y^* \in F(x^*)$ it follows that $y^* \in C(J_{(p(x^*))})$. Let $J^* = J_{p(x^*)}$. Then there exists nonnegative numbers μ_j^*, $j \in J^*$, summing up to one, satisfying

$$y^* = \sum_{j \in J^*} \mu_j^* c^j.$$

Hence,

$$\sum_{j\in J^*} \mu_j^* c^j = \sum_{i\in I} \lambda_i^* a^i + \sum_{h\in I_{n-l}} \nu_h^* d^h.$$

Now, suppose that $x^* \notin \mathrm{bnd}(P')$. Then none of the constraints ${a^i}^\top x = \alpha_i + 1$ is binding at x^* and from the primal-dual theory it follows that $\lambda_i^* = 0$ for all $i \in I$. Hence, we obtain

$$\sum_{j\in J^*} \mu_j^* c^j = \sum_{h\in I_{n-k}} \nu_h^* d^h.$$

Since $C(J) \cap V = \{\underline{0}\}$, this implies that

$$\sum_{j\in J^*} \mu_j^* c^j = \underline{0}.$$

Moreover, by definition of J^*, $p(x^*) \in \cap_{j\in J^*} C^j$ and therefore $\cap_{j\in J^*} C^j \neq \emptyset$. Hence the theorem holds with $T^* = \{h \in J^* \mid \mu_h^* > 0\}$. Finally, suppose that $x^* \in \mathrm{bnd}(P')$. Let

$$I^* = \left\{ i \in I \mid {a^i}^\top x^* = \alpha_i + 1 \right\}.$$

Since $I^* \subset I(x^*)$ and therefore $A(I^*) \subset A(I(x^*))$, we obtain $y^* \in A(I(x^*))$. Since $y^* \in F(x^*)$, we also have that $y^* \in A^*(I(x^*))$. Hence, $y^* \in A(I(x^*)) \cap A^*(I(x^*)) = \{\underline{0}\}$ and thus we have both

$$\sum_{j\in J^*} \mu_j^* c^j = \underline{0}$$

and $p(x^*) \in \cap_{j\in J^*} C^j$. Again the theorem holds with $T^* = \{h \in J^* \mid \mu_h^* > 0)\}$. Q.E.D.

Notice that the condition $C(J) \cap V = \{\underline{0}\}$ is satisfied if $C(J) \subset V^*$.

4 Applications on the unit simplex

In this section we apply the Main Theorem to prove several well-known intersection results on the unit simplex $S^n = \{x \in \mathbf{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$. For $h \in I_n$, S_h^n denotes the facet $S_h^n = \{x \in S^n \mid x_h = 0\}$, and for $T \subset I_n$, $S^n(T) = \cap_{h\in T} S_h^n$. Furthermore, for $S \subset I_n$, let the n-vector m^S be defined by $\sum_{i\in S} \frac{1}{|S|} e^i$, where $|S|$ denotes the number of elements in S and where e^i is the ith unit vector in $\mathbf{R}^n$. Observe that $m^S = e^i$ if $S = \{i\}$. For ease of notation we write $m^{I_n} = m$. Now, take $l = n-1$, $d^1 = m$, $\delta_1 = 1/n$, $I = I_n$, $a^i = m - e^i$ and $\alpha_i = 1/n$ for $i \in I_n$. Observe that $a^i \in V^*$ for all $i \in I_n$. Now, S^n can be rewritten as the polytope P given by

$$S^n = \left\{ x \in \mathbf{R}^n \mid {a^i}^\top x \leq \alpha_i,\ i \in I_n \text{ and } {d^1}^\top x = \delta_1 \right\}.$$

Notice that for $T \subset I_n$, $F(T) = S^n(T)$. The next lemma can already be found in Sperner [16]. The formulation below is due to Fan [1], see also Scarf [14] and Freidenfelds [2].

Theorem 4.1 Sperner Lemma.
Let $\{C^j \mid j \in I_n\}$ be a collection of closed sets covering the unit simplex S^n satisfying that for every $h \in I_n$ the facet S^n_h is a subset of C^h. Then $\cap_{j\in I_n} C^j \neq \emptyset$.

Proof.
Take $J = I_n$ and for $j \in I_n$, take $c^j = -a^j$. Clearly, $C(J) \subset V^*$ and $\underline{0} \in C(J)$. Since the sets C^j, $j \in I_n$ are closed, it follows from the boundary condition that $I_x \subset J_x$ if $x \in \text{bnd}(S^n)$. Hence, if $x \in \text{bnd}(S^n)$, then $C(J_x) \cap A^*(I_x) \neq \emptyset$. For instance take $b = -\sum_{i\in T} \frac{1}{|T|} a^i$, where $T = I_x \neq I_n$. Then $b \in A^*(T) \cap C(T) \subset A^*(I_x) \cap C(J_x)$, since $T \subset J_x$. Since $S^n(T)$ is empty if $T = I_n$, the boundary conditions of the Main Theorem are satisfied and hence there is a balanced set $T^* \subset I_n$ for which $\cap_{j\in T^*} C^j \neq \emptyset$. However, by definition of the vectors c^j, $j \in I_n$, the set I_n is the unique balanced set.
Q.E.D.

The lemma of Sperner is a special case of a theorem of Scarf [13]. Also this theorem follows from the Main Theorem. To state the theorem, let B be an $n \times k$ matrix, $k > n$, satisfying that $b^i = e^i$, $i \in I_n$, with b^j the jth column of B, $j \in I_k$.

Theorem 4.2 Scarf Lemma.
Let $\{C^j \mid j \in I_k\}$ be a collection of closed sets covering the unit simplex S^n satisfying that for every $h \in I_n$ the facet S^n_h is a subset of C^h. Let $c \in \mathbf{R}^n_+ \setminus \{\underline{0}\}$ be given and assume that the set of solutions $\{y \in \mathbf{R}^k_+ \mid By = c\}$ is nonempty and bounded. Then there exists a set $T^* \subset I_k$ such that the system of equations $\sum_{j\in T^*} \mu_j b^j = c$ has a positive solution and $\cap_{j\in T^*} C^j \neq \emptyset$.

Proof.
Take $J = I_k$ and for $j \in I_k$, take $c^j = b^j - \gamma_j c$, where $\gamma_j = nm^\top b^j$. So $\gamma_j = 1$ for $j \in I_n$. Without loss of generality we may assume that $c \in S^n$. Then, $\underline{0} \in C(J)$ and $m^\top c^j = 0$, $j \in I_k$, and hence $C(J) \subset V^*$. Let $x \in \text{bnd}(S^n)$ be such that $x \in F(T)$ with $T = I_x$. From the boundary condition we have that $C(T) \subset C(J_x)$. So the boundary condition of the Main Theorem is satisfied if $A^*(T) \cap C(T) \neq \emptyset$. First suppose that $c_h = 0$ for all $h \in T$. Then, for every $i, j \in T$, it holds that $c^{j\top} a^i = (e^j - c)^\top (m - e^i) = -e^{j\top} e^i \leq 0$, so that $c^j \in A^*(T) \cap C(T)$ for any $j \in T$. Now suppose that $c_h > 0$ for some $h \in T$. Then take $y = \sum_{j\in T} \frac{c_j}{\sum_{h\in T} c_h} c^j$. Clearly, $y \in C(T)$. Moreover, for every $i \in T$, $y^\top a^i = -y_i \leq 0$, and so $y \in A^*(T)$. Hence, the conditions of the Main Theorem are satisfied and there is a balanced set $T^* \subset I_k$ satisfying $\cap_{j\in T^*} C^j \neq \emptyset$. Balancedness of T^* implies that there exist $\lambda^*_j > 0$ for $j \in T^*$ satisfying $\sum_{j\in T^*} \lambda^*_j = 1$ such that $\sum_{j\in T^*} \lambda^*_j c^j = \underline{0}$, i.e, $\sum_{j\in T^*} \lambda^*_j b^j - \alpha^* c = \underline{0}$ with $\alpha^* = \sum_{j\in T^*} \lambda^*_j \gamma_j$. Since $b^i = e^i$ for $i \in I_n$, we have that $c = \sum_{i\in I_n} c_i b^i$, from which it follows that $\sum_{j\in T^*} \lambda^*_j b^j - \sum_{i\in I_n} \alpha^* c_i b^i = \underline{0}$. From the boundedness of the set $\{y \in \mathbf{R}^k_+ \mid By = c\}$ it follows that $\underline{0} \notin \{By \mid y \in \mathbf{R}^k_+ \setminus \{\underline{0}\}\}$. Hence $\alpha^* > 0$ and $\sum_{j\in T^*} \mu^*_j b^j = c$, where $\mu^*_j = \frac{\lambda^*_j}{\alpha^*} > 0$ for all $j \in T^*$. Q.E.D.

The following lemma is the well-known lemma of Knaster, Kuratowski and Mazurkiewicz [9] and can be seen as the dual of the Sperner lemma.

Theorem 4.3 KKM Lemma.
Let $\{C^j \mid j \in I_n\}$ be a collection of closed sets covering the unit simplex S^n satisfying that for every $T \subset I_n$, the face $S^n(T)$ is contained in $\cup_{j \notin T} C^j$. Then $\cap_{j \in I_n} C^j \neq \emptyset$.

Proof.
Take $J = I_n$ and for $j \in I_n$, take $c^j = a^j$. Again, $C(J) \subset V^*$ and $\underline{0}$ belongs to the convex hull of the vectors c^j, $j \in I_n$. From the boundary condition we have that, $J_x \cap I_n \setminus T \neq \emptyset$ if x is a point in the boundary of S^n such that $I_x = T$. Since for any pair a^i and a^j, $i \neq j$, it holds that $a^{i\top} a^j < 0$, we have that for every $T \subset I_n$, $a^j \in A^*(T)$ if $j \notin T$. According to the boundary condition, J_x contains at least one element $j \notin I_x$. Hence $C(J_x) \cap A^*(I_x) \neq \emptyset$ if $x \in \text{bnd}(S^n)$ and the boundary condition of the Main Theorem is satisfied. So, there is a balanced set $T^* \subset I_n$ for which $\cap_{j \in T^*} C^j \neq \emptyset$. Again the set I_n is the unique balanced set. Q.E.D.

The next well-known intersection theorem on the unit simplex is the intersection theorem of Shapley [15]. To do so, we need the concept of balancedness of sets. To distinguish the balancedness of sets from the notion of balancedness of vectors in Definition 2.1, we speak in the former case about set-balancedness.

Definition 4.4
Let $\mathcal{N}$ be the collection of all nonempty subsets of the set of integers I_n. Then a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of k elements of $\mathcal{N}$ is set-balanced if there exist positive numbers λ_j^*, $j = 1, \ldots, k$, such that $\sum_{j=1}^k \lambda_j^* m^{B_j} = m$.

Theorem 4.5 Shapley Lemma.
Let $\{C^S \mid S \in \mathcal{N}\}$ be a collection of closed sets covering the unit simplex S^n satisfying that for every $T \subset I_n$, it holds that $S^n(T) \subset \cup_{S \subset I_n \setminus T} C^S$. Then there is a set-balanced family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of elements of $\mathcal{N}$ for which $\cap_{j=1}^k C^{B_j} \neq \emptyset$.

Proof.
Take $J = \mathcal{N}$ and for any set $S \in \mathcal{N}$, take $c^S = m - m^S$. Then $C(J) \subset V^*$ and $\underline{0} \in C(J)$. From the boundary condition we have that J_x contains some set $S \subset I_n \setminus T$ if $T = I_x$. Moreover, for any $T \subset I_n$, $c^{S\top} a^j \leq 0$ for every $j \in T$ and $S \subset I_n \setminus T$. Hence, for any x on the boundary of S^n, $C(J_x) \cap A^*(I_x) \neq \emptyset$. From the Main Theorem it follows then that there exists a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ satisfying that the collection of vectors c^{B_j}, $j = 1, \ldots, k$, is balanced and $\cap_{j=1}^k C^{B_j} \neq \emptyset$. By definition of the vectors c^S, $S \in \mathcal{N}$, the family $\mathcal{B}$ is set-balanced and hence the theorem holds. Q.E.D.

The following intersection theorem on the unit simplex is due to Ichiishi [6] and can be considered as the dual of the Shapley Lemma.

Theorem 4.6 Ichiishi Lemma.
Let $\{C^S \mid S \in \mathcal{N}\}$ be a collection of closed sets covering the simplex S^n satisfying that for every $T \subset I_n$ it holds that $S^n(T) \subset \cup_{T \subset S} C^S$. Then there is a set-balanced family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of elements of $\mathcal{N}$ for which $\cap_{j=1}^k C^{B_j} \neq \emptyset$.

Proof.
For $J = \mathcal{N}$, take $c^S = m^S - m$ for every set $S \in \mathcal{N}$. Then again $C(J) \subset V^*$ and $\underline{0} \in C(J)$. From the boundary condition we have that J_x contains some set S having T as a subset if $x \in S^n(T)$. Moreover, for any $T \subset I_n$, $c^{S\top} a^j \leq 0$ for every $j \in T$ and $S \supset T$. Hence, the boundary condition of the Main Theorem is satisfied and there exists a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ satisfying that the collection of vectors c^{B_j}, $j = 1, \ldots, k$, is balanced and $\cap_{j=1}^k C^{B_j} \neq \emptyset$. By definition of the vectors c^S, then also the family $\mathcal{B}$ is set-balanced and hence the theorem holds. Q.E.D.

In the next two lemma's the unit simplex S^n is covered by a collection of sets C^{ij}, $i, j \in I_n$. The second lemma is established by Gale [5] and generalizes the KKM lemma by imposing for every $j \in I_n$ the KKM boundary conditions for the collection $\{C^{ij} \mid i \in I_n\}$. To prove this lemma by using the Main Theorem we have to reformulate the problem on the product space of two unit simplices. This approach is very similar to the proof given in Ichiishi and Idzik [7]. The first lemma is due to Joosten and Talman [8] and can be proved directly by applying the Main Theorem.

Theorem 4.7 Joosten and Talman Lemma.
Let $\{C^{ij} \mid i \in I_n, j \in I_n\}$ be a collection of closed sets covering the unit simplex S^n satisfying that if $x \in bnd(S^n)$, then $x \in C^{ij}$ for some $j \in I_n$ with $x_j > 0$ or $x \in \cup_{j \in I_n} C^{ij}$ for all $i \in I_n$ for which $x_i = 0$. Then there exists a nonempty set $I^* \subset I_n$ and, with $k = |I^*|$, a permutation $\sigma(I^*) = (\sigma_1, \ldots, \sigma_k)$ of the elements of I^*, such that $\cap_{i \in I^*} C^{i\sigma_i} \neq \emptyset$.

Proof.
Take $J = I_n \times I_n$ and for $(i, j) \in J$, take $c^{ij} = e^i - e^j$. Clearly, $C(j) \subset V^*$ and $\underline{0} \in C(J)$. Without loss of generality we may assume that $C^{ii} = \emptyset$ for every $i \in I_n$. Suppose, there is an $x \in \text{bnd}(S^n)$ such that $x_j = 0$ if $x \in C^{ij}$. Because of the boundary condition we then have $x \in \cup_{j \in I_n} C^{ij}$ for all $i \in I_n$ for which $x_i = 0$. As shown in [8] then there exists a set $I^* \subset \{i \in I_n \mid x_i = 0\}$ and a permutation $\sigma(I^*)$ such that $x \in \cap_{i \in I^*} C^{i\sigma_i}$ and hence the theorem holds. Otherwise, for every $x \in \text{bnd}(S^n)$, $x \in C^{ij}$ for some pair $(i, j) \in J$ with $x_j > 0$. Then, for such a pair (i, j), $(c^{ij})^\top a^h \leq 0$ for every $h \in I_x$ since $j \neq h$. Hence, $A^*(I_x) \cap C(j_x) \neq \emptyset$ if $x \in \text{bnd}(S^n)$. So, the boundary condition of the Main Theorem is satisfied and there is a balanced set $T^* \subset J$ for which $x \in \cap_{(i,j) \in T^*} C^{ij} \neq \emptyset$. Clearly, by definition of the vectors c^{ij}, $(i, j) \in J$, balancedness of T^* implies that there exists a set $I^* \subset I_n$ and a permutation $\sigma(I^*)$ such that $\{(i, \sigma_i) \mid i \in I^*\} \subset T^*$. Q.E.D.

Theorem 4.8 Gale Lemma.
For each $j \in I_n$, let $\{C^{ij} \mid i \in I_n\}$ be a collection of closed sets covering the simplex S^n and satisfying that for every $T \subset I_n$, $S^n(T) \subset \cup_{i \notin T} C^{ij}$. Then there exists a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the indices of I_n, such that $\cap_{j \in I_n} C^{\sigma_j j} \neq \emptyset$.

Proof.
To prove this theorem, we apply the Main Theorem on the simplotope S given by $S = S^n \times S^n$, i.e.,

$$S = \left\{ x \in \mathbf{R}^{2n} \mid \tilde{a}^{i\top} x \leq \alpha_i,\ i \in I_{2n} \text{ and } \tilde{d}^{h\top} x = \delta_1,\ h = 1, 2 \right\}$$

with $\tilde{d}^1 = (d^{1\top}, \underline{0}^\top)^\top$, $\tilde{d}^2 = (\underline{0}^\top, d^{1\top})^\top$, $\tilde{a}^i = (a^{i\top}, \underline{0}^\top)^\top$, $i \in I_n$, $\tilde{a}^i = (\underline{0}^\top, a^{(i-n)\top})^\top$, $i \in I_{2n} \setminus I_n$. Moreover, define the $2n$-dimensional vectors c^{ij} by $c^{ij} = (a^{i\top}, a^{j\top})^\top$, $i, j \in I_n$. Observe that $C(J) \subset V^*$ and $\underline{0} \in C(J)$. Finally, define the sets $\tilde{C}^{ij} \subset S^n \times S^n$ by $\tilde{C}^{ij} = C^{ij} \times S^n$. Now, the covering $\{\tilde{C}^{ij},\ i, j \in I_n\}$ of S satisfies the boundary condition of the Main Theorem. To see this, for two nonempty subsets $T_1, T_2 \subset I_n$, let x be a point in the relative interior of the face $S^n(T_1) \times S^n(T_2)$ of $S^n \times S^n$. Then, for every $i \notin T_1$ and $j \notin T_2$, $c^{ij\top} \tilde{a}^h < 0$ for all $h \in T_1$ and $h - n \in T_2$ and hence $c^{ij} \in A^*(T_1 \cup \bar{T}_2)$, where $\bar{T}_2 = \{h \in I_{2n} \setminus I_n \mid h - n \in T_2\}$. Moreover, by the boundary condition we have that for any $j \in I_n$, there exists an $i \notin T_1$ such that $x \in \tilde{C}^{ij}$ and hence $c^{ij} \in C(J_x)$. Thus, for $j \notin T_2$ and $i \notin T_1$, we have that $c^{ij} \in C(J_x) \cap A^*(T_1 \cup \bar{T}_2)$. Since $I_x = T_1 \cup \bar{T}_2$ the boundary condition on S is satisfied. Hence there is a balanced collection of vectors c^{ij} such that the corresponding sets $\tilde{C}^{ij}$ have a non-empty intersection. So, there are a set $T \subset I_n \times I_n$ and positive numbers μ_{ij}, $(i, j) \in T$ such that

$$\sum_{(i,j) \in T} \mu_{ij} c^{ij} = \underline{0}$$

and

$$\cap_{(i,j) \in T} \tilde{C}^{ij} \neq \emptyset.$$

Define $\mu^*_{ij} = \frac{\mu_{ij}}{\sum_{hk} \mu_{hk}}$. Since $c^{ij} = \tilde{a}^i + \tilde{a}^{j+n}$, it follows that for each $i \in I_n$ we must have that $\sum_j \mu^*_{ij} = \frac{1}{n}$ and that for each $j \in I_n$, $\sum_i \mu^*_{ij} = \frac{1}{n}$. From this property it follows that the $n \times n$ matrix U defined by $u_{ij} = n\mu^*_{ij}$ if $(i, j) \in T$ and $u_{ij} = 0$ if $(i, j) \notin T$ is a double stochastic matrix and therefore U is a convex combination of permutation matrices according to the theorem of Birkhoff and von Neumann. So, there exists a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the indices of I_n, such that $u_{\sigma_j, j} > 0$ and hence $(\sigma_j, j) \in T$. Since $\cap_{(i,j) \in T} \tilde{C}^{ij} \neq \emptyset$ implies that $\cap_{(i,j) \in T} C^{ij} \neq \emptyset$, this proves the theorem. Q.E.D.

In the Gale Lemma we have that for each $j \in I_n$ the collection of sets $\{C^{ij} \mid i \in I_n\}$ satisfies the condition of the KKM Lemma. In the following theorem we have that for each $j \in I_n$ the collection of sets $\{C^{ij} \mid i \in I_n\}$ satisfies the condition of the Sperner Lemma. So, this theorem can be considered as the dual of the Gale Lemma. The proof goes along the same lines of Theorem 4.8 and is left to the reader.

Theorem 4.9
For each $j \in I_n$, let $\{C^{ij} \mid i \in I_n\}$ be a collection of closed sets covering the simplex S^n and satisfying that for every $h \in I_n$, the facet S^n_h is a subset of C^{hj}. Then there exists a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the indices of I_n, such that $\cap_{j \in I_n} C^{\sigma_j j} \neq \emptyset$.

5 Intersection theorems on the polytope

In this section the Main Theorem is applied to generalize several of the intersection theorems on the unit simplex to an arbitrary polytope P. In these generalizations without loss of generality we will ignore the equality constraints on P. Hence $V = \{\underline{0}\}$, and $C(J) \cap V = \{\underline{0}\}$ is satisfied if $\underline{0} \in C(J)$. Now, the polytope under consideration equals

$$P = \left\{x \in \mathbf{R}^n \mid a^{i\top} x \leq \alpha_i,\, i \in I\right\}$$

with I a set of at least $n+1$ elements. Since P is bounded we have that $\underline{0}$ lies in the convex hull of the vectors a^j, $j \in I$. Now, we first generalize the Sperner Lemma to the polytope. To prove this generalization, we state the following lemma, in which for $T \subset I$, $\tilde{A}(T) = \{x \in \mathbf{R}^n \mid x = \sum_{i \in T} \lambda_i a^i,\ \sum_{i \in T} \lambda_i = 1,\ \lambda_i \geq 0,\ i \in T\}$.

Lemma 5.1 *For any $T \subset I$, $\tilde{A}(T) \cap -A^*(T) \neq \emptyset$.*

<u>Proof.</u>
Suppose $\underline{0} \in \tilde{A}(T)$. Then the lemma holds with $y = \underline{0}$. Now, suppose that $\underline{0} \notin \tilde{A}(T)$. Since $\tilde{A}(T) \subset A(T)$ and $A(T) \cap A^*(T) = \{\underline{0}\}$, it follows that $\tilde{A}(T) \cap A^*(T) = \emptyset$. Hence, for every $x \in \tilde{A}(T)$, $x^\top a^i > 0$ for at least one $i \in T$. Therefore, $\alpha^* > 0$ with α^* the optimal value of the primal linear programming problem

$$\text{(P)} \quad \min \alpha, \text{ s.t. } x \in \tilde{A}(T) \text{ and } \alpha \geq x^\top a^i, \text{ for all } i \in T.$$

Now, let ν^* be the optimal value of the dual linear programming problem

$$\text{(D)} \quad \max \nu, \text{ s.t. } -y \in \tilde{A}(T) \text{ and } \nu \leq -y^\top a^i, \text{ for all } i \in T.$$

Then according to the primal-dual theory we have that $\nu^* = \alpha^* > 0$. Let y^* be any solution to (D). Then, $-y^{*\top} a^i \geq \nu^* > 0$ for all $i \in T$ and hence $y^* \in A^*(T)$. Since $-y^* \in \tilde{A}(T)$, this proves the lemma. Q.E.D.

We can now prove the following generalization of Sperner's lemma.

Theorem 5.2
Let $\{C^j \mid j \in I\}$ be a collection of closed sets covering the polytope P satisfying that for every $h \in I_m$ the facet F_h is a subset of C^h. Then there exist a set $T^* \subset I_m$ and positive numbers μ_j, $j \in T^*$, such that $\{a^j \mid j \in T^*\}$ is balanced and $\cap_{j\in T^*} C^j \neq \emptyset$.

Proof.
Take $J = I$ and $c^h = -a^h$, $h \in I$. Clearly, $\underline{0} \in C(J)$. For a boundary point x in the relative interior of $F(T)$, we have that $-\tilde{A}(T) \subset C(J_x)$ since $x \in C^h$ for every $h \in T$. From Lemma 5.1 it follows that there exists a $y \in -\tilde{A}(T)$ satisfying $y \in A^*(T)$. So, $y \in C(J_x) \cap A^*(I_x)$, since $-\tilde{A}(T) \subset C(J_x)$ and $I_x = T$. Hence the boundary condition of the Main Theorem is satisfied and there exists a balanced set $T^* \subset I_m$ for which $\cap_{j\in T^*} C^j \neq \emptyset$. Clearly, if T^* is balanced, then also $\{a^j \mid j \in T^*\}$ is balanced. Q.E.D.

Theorem 5.2 says that under the boundary condition there exists a set of indices T^* such that the intersection of the sets C^h, $h \in T^*$, is not empty and $\underline{0}$ lies in the interior of the convex hull of the vectors a^h, $h \in T^*$. The theorem is illustrated in Figure 1 for $n = 2$ and $I = I_5$. In this figure x^* is the only intersection point. At x^* it holds that $x^* \in C^1 \cap C^2 \cap C^4$ whereas $\underline{0}$ lies in the relative interior of the convex hull of a^1, a^2, a^4 and hence the set $\{1, 2, 4\}$ is balanced. Notice that the points x^1 and x^2 do not satisfy the requirements because at x^1 we do not have that $\underline{0}$ lies in the convex hull of a^2, a^3, a^4 and at x^2 not that $\underline{0}$ lies in the convex hull of a^1, a^4, a^5.

We remark that in case P is an n-dimensional simplex S given by $S = \{x \in \mathbf{R}^n \mid a^{i\top} x \leq b_i \text{ for } i \in I_{n+1}\}$, we have that the polytope is covered by $n + 1$ closed sets. Since for any $k \leq n$, $\underline{0}$ can not lie in the relative interior of the convex hull of any k of the a^j's, we must have that $T^* = I_{n+1}$ and hence by the Main Theorem all $n + 1$ sets have a nonempty intersection. This result is equivalent to Sperner's intersection theorem on the (unit) simplex, see Theorem 4.1. In general, if in Theorem 5.2 the set T^* contains more than $n + 1$ elements, then the set T^* can be reduced to a set T' of $n + 1$ indices, such that the vectors a^j, $j \in T'$, are affinely independent. Notice that we do not require that in Theorem 5.2 the set T^* must exist of precisely $n + 1$ elements. For example, let the polytope be the n-dimensional unit cube K^n given by

$$K^n = \{x \in \mathbf{R}^n \mid 0 \leq x_i \leq 1 \text{ for } i \in I_n\}.$$

Let us denote the $2n$ facets of K^n by $F_{+i} = \{x \in K^n \mid x_i = 1\}$ and $F_{-i} = \{x \in K^n \mid x_i = 0\}$, for $i = 1, \ldots, n$. In this case $I = I_n \cup -I_n$ and the polytope is covered by $2n$ closed sets, denoted by C^{+i} and C^{-i} for $i = 1, \ldots, n$, such that C^h contains F_h for every $h \in I_n \cup -I_n$. Under this boundary condition we have the following corollary.

Corollary 5.3
Let P be the n-dimensional unit cube K^n and let $\{C^h \mid h \in I_n \cup -I_n\}$ be a collection of closed sets covering K^n such that for every $h \in I_n \cup -I_n$ the facet F_h is a subset of C^h. Then there is an index $i \in I_n$ such that

$$C^{+i} \cap C^{-i} \neq \emptyset.$$

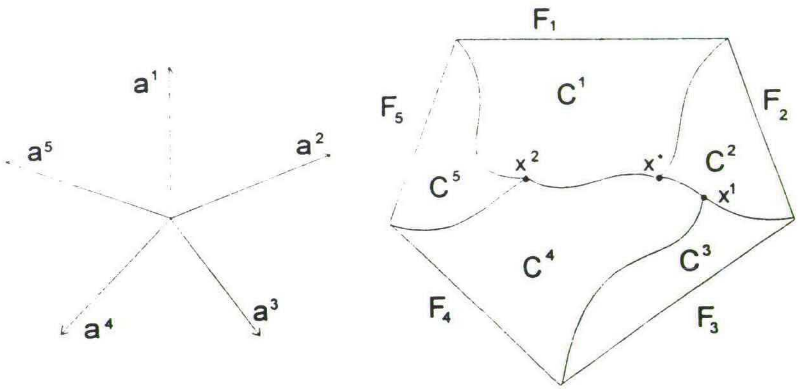

Figure 1: Illustration of Theorem 5.2; n = 2, $I = I_5$

Proof.
We have that $a^{+j} = e^j$ and $a^{-j} = -e^j$ for every $j \in I_n$. But for any $T \subseteq I_n \cup -I_n$, the system $\sum_{h \in T} \nu_h a^h = 0$ can only have a positive solution if for some $i \in I_n$ both $+i$ and $-i$ belong to T. Q.E.D.

Corollary 5.3 is illustrated in Figure 2 for n = 2. In this figure the whole curve between x^1 and x^2 is the intersection of C^{+2} and C^{-2}.

In Theorem 5.2 it is assumed that for every $h \in I$ the facet F_h lies in C^h. The next corollary follows immediately from Theorem 5.2 by taking the sets covering P equal to $C^j \cup F_j$, $j \in I_m$.

Corollary 5.4
Let $\{C^j \mid j \in I\}$ be an arbitrary collection of closed sets covering the polytope P. Then there exist a set $T^* \subset I$ and an $x^* \in P$, such that $\{a^j \mid j \in T^*\}$ is balanced and for every $j \in T^*$, $x^* \notin C^j$ implies $x^* \in F_j$.

We remark that in Corollary 5.4 we allow that some of the sets C^j are empty. In particular, suppose that P is covered by just one set C^k for some $k \in I_m$. Now, let x^* be a solution to the linear programming problem

$$\min x^\top a^k \text{ subject to } a^{i\top} x \leq b_i \text{ for all } i \in I \qquad (1)$$

and let K be the set of indices given by $K = \{i \in I \mid a^{i\top} x^* = b_i\}$. Then it follows from the dual of (1) that there exist a set $J^* \subset K$ and positive numbers ν_i^* for $i \in J^*$ such

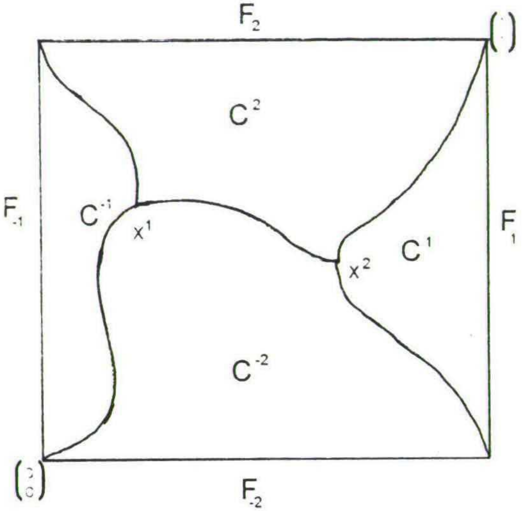

Figure 2: Illustration of Corollary 5.3; n = 2

that $\sum_{i \in J^*} \nu_i^* a^i = -a^k$ and hence with $\nu_k^* = 1$ we have that $T^* = J^* \cup \{k\}$ satisfies the conditions with x^* in the intersection of the sets $C^j \cup F_j$, $j \in T^*$. Corollary 5.4 is illustrated in Figure 3 for $n = 2$ and $I = I_5$. In this figure the sets C^1 and C^2 are empty and the point x^* is the only intersection point. It lies in $F_1 \cap F_2 \cap C^4$, whereas $\underline{0}$ lies in the relative interior of the convex hull of a^1, a^2 and a^4, and hence $T^* = \{1, 2, 4\}$ is balanced. Notice that the set $\{3, 4, 5\}$ is not balanced and hence x^1 is not an intersection point.

Theorem 5.2 can be seen as the continuous analog of a theorem of Freund [3], who shows that in a labelled simplicial subdivision of the polytope P with label set I, there exists a set T of labels, such that $\underline{0}$ lies in the convex hull of the set $\{a^h,\ h \in T\}$ and there is a simplex σ such that the set T is the set of labels of the vertices of σ. This theorem of Freund contains as a special case a result of van der Laan and Talman [10], implying that in any properly labelled simplicial subdivision of an n-dimensional cube K^n with label set $I_n \cup -I_n$, there exist an integer $h \in I_n$ and a simplex σ having two vertices with labels h and $-h$. This result is the combinatorial analog of Corollary 5.4.

In the next theorem we generalize the KKM lemma on the unit simplex to the polytope P. To do so, we first observe that the face $S^n(T)$ of the unit simplex S^n is just the convex hull of the vertices e^i, $i \notin T$. Hence $S^n(T)$ can be written as $\Delta^{I_n \setminus T}$ with Δ^S defined as the convex hull of the vertices e^i, $i \in S$. Doing so, the boundary condition of the KKM lemma can be rewritten as $\Delta^S \subset \cup_{j \in S} C^j$ for

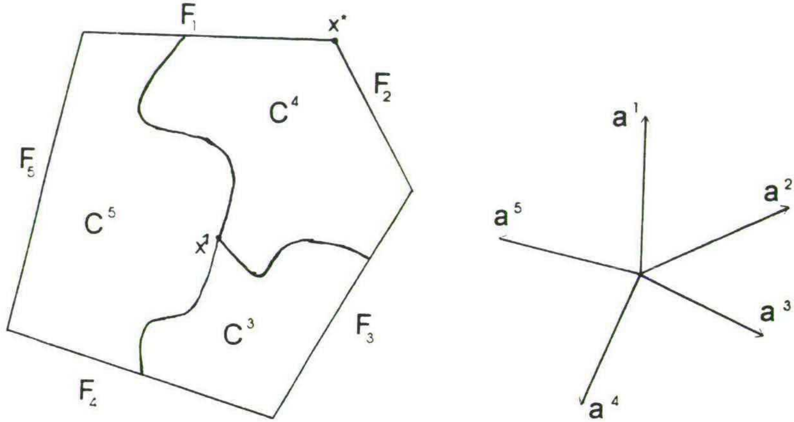

Figure 3: Illustration of Corollary 5.4; n = 2, $I = I_5$

any $S \subset I_n$. This gives an important insight in the difference between the Sperner lemma and the KKM lemma. In the Sperner lemma, each facet S^n_h is covered by C^h and so the number of sets C^h is equal to the number of facets. In generalizing the Sperner lemma to the polytope, we indeed had that in Theorem 5.2 the number of sets covering the polytope equals the number of constraints. However, in the KKM lemma we have that each face, being the convex hull of a number of vertices, is covered by the union of sets corresponding to the vertices carrying the face. So, the number of sets equals the number of vertices. For $n = 2$ or $|I| = n + 1$ the number of vertices equals the number of facets. However, for $n > 2$ and $|I| > n + 1$, this is in general not the case. So, generalizing the KKM lemma to the polytope, the number of sets covering P should be equal to the number of vertices. Therefore, for a given polytope $P = \{x \in \mathbf{R}^n \mid a^{i\top} x \leq \alpha_i,\ i \in I\}$, let t be the number of vertices and let $\{v^j,\ j \in I_t\}$ be the set of vertices of P. Then we have the following generalization of the KKM lemma, in which the boundary condition says that every face is covered by the sets labelled by the vertices of that face.

Theorem 5.5

Let $\{C^j \mid j \in I_t\}$ be a collection of closed sets covering the polytope P with vertices v^j, $j \in I_t$, satisfying that every face $F(T)$, $T \subset I$, is covered by $\cup\{C^h \mid v^h \in F(T)\}$. Then for any $c \in P$ there exists a set $T^* \subset I_t$ such that c lies in the convex hull of the vectors v^j, $j \in T^*$, and $\cap_{j \in T^*} C^j \neq \emptyset$.

Proof.
Take $J = I_t$ and for $j \in J$, take $c^j = c - v^j$. Clearly, $\underline{0} \in C(J)$. Let x be a boundary point of P and let v^h be a vertex of $F(T)$, where $T = I_x$, i.e., x lies in the relative interior of $F(T)$. Then we have that $v^{h\top} a^i = \alpha_i$ for any $i \in T$. Hence, for every $i \in T$ it holds that $c^{h\top} a^i \leq 0$ because $c^\top a^i \leq \alpha_i$. Thus, for any vertex v^h of $F(T)$ we have that $c^h \in A^*(T)$. Moreover, by the boundary condition, $x \in C^h$ for at least one set C^h corresponding to a vertex v^h of $F(T)$ and therefore $c^h \in C(J_x)$ for at least one v^h in $F(T)$. Hence, the boundary condition of the Main Theorem is satisfied and there exists a balanced set $T^* \subset J$ for which $\cap_{j \in T^*} C^j \neq \emptyset$. Clearly, the balancedness of T^* implies that c lies in the convex hull of the vertices v^j, $j \in T^*$. Q.E.D.

Observe that in case of the KKM lemma on the unit simplex we have that $v^h = e^h$, $h \in I_n$. Taking $c = m \in S^n$ we obtain that $c^h = m - e^h$, showing that Theorem 5.5 contains Theorem 4.3 as a special case.

Observe that Theorem 5.5 holds for any $c \in P$. If $\underline{0} \in P$, we may take $c = \underline{0}$ and obtain the result that there is a set of indices T^* for which the corresponding sets C^j have a nonempty intersection and for which the collection of corresponding vertices is balanced. In the same way Theorem 5.2 can be generalized by stating that for any $c \in C(J)$, there exists a set T^* for which the corresponding sets C^j have a nonempty intersection and for which c is contained in the convex hull of the vectors $-a^j$, $j \in T^*$.

Theorem 5.5 is illustrated in the Figures 4 and 5. In Figure 4 the point c lies in the convex hull of v^1, v^3 and v^5, whereas the sets C^1, C^3 and C^5 meet each other in the point x^*. In Figure 5 we have that x^1 is the intersection point if c lies in the interior of the triangle $< v^1, v^2, v^3 >$, x^2 is the intersection point if c lies in the interior of $< v^1, v^3, v^4 >$, while the whole curve between x^1 and x^2 is the set of intersection point if c lies on the line $[v^1, v^2]$.

Notice that Theorem 5.5 is a special case of Theorem 3.1 in Ichiishi and Idzik [7]. It can be shown that under the conditions of Theorem 3.1 of Ichiishi and Idzik the conditions of the Main Theorem are satisfied and therefore the Ichiishi and Idzik Theorem 3.1 also follows from the Main Theorem. Ichiishi and Idzik also show that their Theorem 3.1 is a special case from the theorem below, stated as Theorem 3.4 in [7]. Below we show that also this latter theorem is a special case of our Main Theorem. In the following, aff(S) denotes the affine hull of the set S.

Theorem 5.6 Ichiishi and Idzik Theorem.
For $k > n$, let B be an $n \times k$ matrix with colums b^j, $j \in I_k$, and for t, $n \leq t \leq k$, let W be the convex hull of the vectors b^j, $j \in I_t$. Let $\{C^j \mid j \in I_k\}$ be a collection of closed sets covering the set W. For some $c \in W$, assume that the set of solutions $\{y \in \mathbf{R}^k_+ \mid By = c\}$ is bounded. Furthermore, assume that $\underline{0} \notin \mathit{aff}\,(W)$, $b^j \in \mathit{aff}\,(W \cup \underline{0})$ for all $j \in I_k$, and that for every proper face F of W it holds that $F \subset \cup\{C^j \mid b^j \in \mathit{aff}\,(F) + \mathit{aff}\,\{\underline{0}, c\},\ j \in I_k\}$. Then there exists a set $T^* \subset I_k$ such that the system of equations $\sum_{j \in T^*} \mu_j b^j = c$ has a positive solution and $\cap_{j \in T^*} C^j \neq \emptyset$.

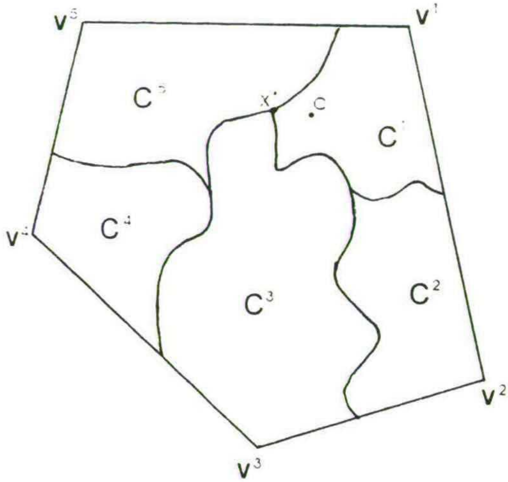

Figure 4: Illustration of Theorem 5.5; n = 2, $I = I_5$

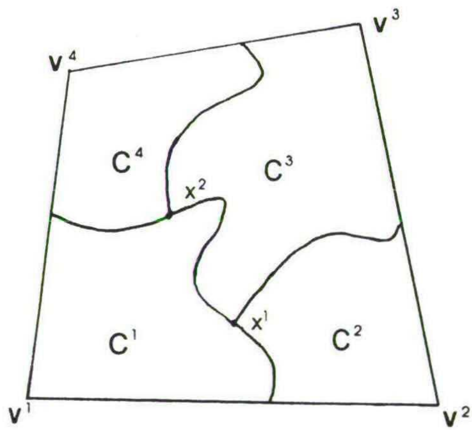

Figure 5: Illustration of Theorem 5.5; n = 2, $I = I_4$

Proof.
Since $\underline{0} \notin W$, without loss of generality we may assume that

$$\text{aff}\,(W) = \{x \in \mathbf{R}^n \mid \sum_{j=1}^{n} x_j = 1\}$$

and hence $l = n - 1$ and $d^1 = m$, so that $V = \{x \in \mathbf{R}^n \mid x = \nu m, \nu \in \mathbf{R}\}$. Now, take $J = I_k$, and for $j \in J$, take $c^j = -b^j + \gamma_j c$, where $\gamma_j = \sum_{i=1}^{n} b_i^j$. Since $c \in W$ we have that $\underline{0} \in C(J)$. Moreover, $m^\top c = 1/n$ and hence for all $j \in I_k$ we have that $m^\top c^j = 0$ and therefore $c^j \in V^*$. So $C(J) \subset V^*$. Now, let x be a point in the interior of some face F of W. Then, $I_x \subset I$ is the set of elements corresponding to its binding constraints, i.e., $F = \{x \in W \mid a^{i\top} x = \alpha_i, i \in I_x\}$, with the appropriate vectors $a^i \in \mathbf{R}^n$ and numbers $\alpha_i \in \mathbf{R}$, $i \in I_x$. According to the boundary condition there exists an element $h \in I_k$ such that both $c^h \in C(J_x)$ and $b^h = d^h + \delta_h c$, for some $d^h \in \text{aff}(F(I_x))$ and $\delta_h \in \mathbf{R}$. By substituting b^h we now obtain that $c^h = -d^h + (\gamma_h - \delta_h)c$. By definition, $\sum_{j=1}^{n} c_j^h = 0$, whereas both d^h and c are in W. Hence, $0 = \sum_{j=1}^{n} c_j^h = \sum_{j=1}^{n} -d_j^h + (\gamma_h - \delta_h)\sum_{j=1}^{n} c_j = -1 + \gamma_h - \delta_h$ and so $c^h = c - d^h$. Moreover, $a^{i\top} c \leq \alpha_i$ for all $i \in I$ and $a^{i\top} d^h = \alpha_i$ for all $i \in I_x$. So, for all $i \in I_x$, $a^{i\top} c^h = a^{i\top} c - a^{i\top} d^h \leq 0$ and thus $c^h \in A^*(I_x)$. Therefore, $c^h \in A^*(I_x) \cap C(J_x)$ and the conditions of the Main Theorem are satisfied. Hence, there exists a balanced set $T^* \in I_k$ such that $\cap_{j \in T^*} C^j \neq \emptyset$. Balancedness of T^* implies that there exist $\lambda_j^* > 0$ for $j \in T^*$ satisfying $\sum_{j \in T^*} \lambda_j^* c^j = \underline{0}$. Hence, $\sum_{j \in T^*} \lambda_j^* b^j = \alpha^* c$ with $\alpha^* = \sum_{j \in T^*} \lambda_j^* \gamma_j$. Since the set of solutions $\{y \in \mathbf{R}_+^k \mid By = c\}$ is bounded, we must have that $\alpha^* > 0$. Hence, $\sum_{j \in T^*} \mu_j^* b^j = c$, where $\mu_j^* = \frac{\lambda_j^*}{\alpha^*} > 0$ for every $j \in T^*$. Q.E.D.

Finally we consider the generalizations of the Shapley and the Ichiishi lemma on the unit simplex to the polytope. The first case is a straightforward generalization of Theorem 5.5 by allowing that the polytope P is covered by a collection of closed subsets $\{C^S \mid S \in \mathcal{L}\}$, where $\mathcal{L}$ is the collection of nonempty subsets of I_t. For $S \in \mathcal{L}$, let v^S be any point satisfying that $v^{S\top} a^i \geq \alpha_i$ for every $i \in T$, where T is such that $v^{h\top} a^i = \alpha_i$ for all $h \in S$. Then we have the following result.

Theorem 5.7
Let $\{C^S \mid S \in \mathcal{L}\}$ be a collection of closed sets covering the polytope P with vertices v^h, $h \in I_t$, satisfying that every face $F(T)$, $T \subset I_m$, is covered by $\cup\{C^S \mid v^h \in F(T)$ for all $h \in S\}$. Then, for any $c \in P$, there exists a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of k elements of $\mathcal{L}$ such that c lies in the convex hull of the vectors v^{B_j}, $j = 1, \ldots, k$, and $\cap_{j=1}^{k} C^{B_j} \neq \emptyset$.

Proof.
Take $J = \mathcal{L}$ and for $S \in \mathcal{L}$, take $c^S = c - v^S$. Again, $\underline{0} \in C(J)$. Let x be a point in the interior of the face $F(T)$, so $I_x = T$. Then there exists an $S \in J_x$ such that $v^h \in F(T)$ for all $h \in S$. Since $v^{S\top} a^i \geq \alpha_i$, we have that $c^{S\top} a^i \leq 0$ for every $i \in T$.

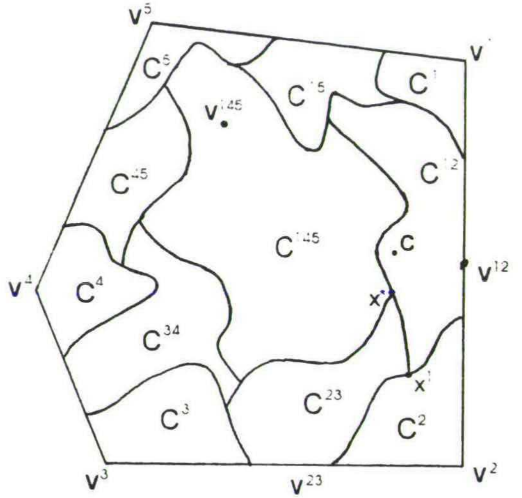

Figure 6: Illustration of Theorem 5.7; n = 2, $I = I_5$

Consequently, $c^S \in A^*(T)$. Hence, the boundary condition of the Main Theorem is satisfied. So there exists a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of elements in $\mathcal{L}$ such that the collection of vectors c^{B_j}, $j = 1, \ldots, k$, is balanced and $\cap_{j=1}^k C^{B_j} \neq \emptyset$. Q.E.D.

Theorem 5.7 is illustrated in Figure 6. In this figure the point c lies in the convex hull of v^{12}, v^{23} and v^{145}, whereas the sets C^{12}, C^{23} and C^{145} meet each other in point x^*. Observe that the point x^1 does not satisfy for the point c chosen in the figure, but becomes an intersection point as soon as c is moved to a point in the convex hull of v^{12}, v^{23} and v^2.

To generalize the lemma of Ichiishi on the unit simplex to the polytope P, let $\mathcal{M}$ be the collection of subsets T of I for which $F(T)$ is a face of P, and for $T \in \mathcal{M}$, let a^T be any point in $\tilde{A}(T) \cap -A^*(T)$. From Lemma 5.1 we know that the latter set is nonempty. Notice that $a^T = a^h$ if $T = \{h\}$ for some $h \in I$.

Theorem 5.8
Let $\{C^S \mid S \in \mathcal{M}\}$ be a collection of closed sets covering the polytope P satisfying that every face $F(T)$, $T \in \mathcal{M}$, is covered by $\cup\{C^S \mid T \subset S\}$. Then there exists a balanced collection of vectors a^{B_j}, $j = 1, \ldots, k$, such that $\cap_{j=1}^k C^{B_j} \neq \emptyset$.

Proof.
Take $J = \mathcal{M}$ and for $S \in \mathcal{M}$, take $c^S = -a^S$. Clearly, $\underline{0} \in C(J)$. Let x be a boundary point of P. Then there exists an $S \in J_x$ such that $T \subset S$ and $x \in C^S$, where

$T = I_x$. Since $T \subset S$, we have that $A^*(S) \subset A^*(T)$ and hence $c^S \in A^*(T)$. Moreover $S \subset J_x$ and hence $c^S \in C(J_x)$. So, the boundary condition of the Main Theorem is satisfied and there exists a family $\mathcal{B} = \{B_1, \ldots, B_k\}$ of k elements in $\mathcal{M}$ such that the collection of vectors c^{B_j}, $j = 1, \ldots, k$, is balanced and $\cap_{j=1}^k C^{B_j} \neq \emptyset$. Since $c^S = -a^S$ for all $S \in \mathcal{M}$, this implies that the collection of vectors $\{a^{B_j} \mid j = 1, \ldots, k\}$ is also balanced.

Q.E.D.

Observe that $a^S \in \mathring{A}(S)$ and hence a^S is the convex combination of the vectors a^h, $h \in S$. This implies that the theorem can be reformulated by stating that there exists a collection $\mathcal{B} = \{B_1, \ldots, B_k\}$ of elements of $\mathcal{M}$ such that $\cap_{j=1}^k C^{B_j} \neq \emptyset$ and $\{a^j \mid j \in T^*\}$ is balanced, where $T^* = \cup_{j=1}^k B_j$. This immediately proves Theorem 5.2, when we take $F(T) = C^T$ in case $|T| > 1$. Finally we remark that the result of Theorem 5.8 still holds when $\mathcal{M}$ equals the collection of all nonempty subsets of I or if a^T is any point in $-A^*(T)$. In the latter case a^T need not be an element in $A(T)$.

In case the polytope is the product space of N unit simplices, i.e., $P = \Pi_{j=1}^N S^{n_j}$, the intersection theorems stated in Talman [17] and van der Laan and Talman [12] are obtained. In the latter paper the intersection theorems of Sperner and KKM on the unit simplex are generalized to the simplotope and are applied in providing a direct proof of the existence of a Nash equilibrium in a noncooperative game with N players with player j having n_j actions. In [17] also the results of Shapley and Ichiishi are generalized to the simplotope.

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